

Linking THEORY and PRACTICE in Teaching GEOMETRY

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Regarding the relationship between theory and practice in teaching mathematics, Anna Sfard (2003) wrote,

All too often, a new promising idea is embraced to the total exclusion of alternative possibilities. In this way, what was intended as but an ingredient becomes the whole meal; what was supposed to be an optional technique for those who find it helpful and pleasing gradually becomes the only legitimate way of doing things. (p. 354)

I will discuss examples of how van Hiele theory proved to be a useful “ingredient” in my teaching of a summer course for high school students who had failed geometry during the school year. The theory provided a framework to help me organize and reflect upon instruction for some key concepts in geometry.

OVERVIEW AND APPLICATION OF VAN HIELE THEORY

Van Hiele’s (1986) research suggests that students progress through a hierarchy of five different levels of thinking as they learn geometry. Burger and Shaughnessy (1986) named the levels (from lowest to highest) holistic, analytic, abstract, deductive, and rigorous. **Table 1** summarizes the characteristics of these levels, as described by Burger and Shaughnessy (1986).

TABLE 1**Summary of the van Hiele Levels, as Described by Burger and Shaughnessy (1986)**

Van Hiele	
Level	Description
Holistic	<ul style="list-style-type: none"> • Students use imprecise properties to compare shapes (e.g., a cookie is like a circle). Comparisons are idiosyncratic. • Students use irrelevant attributes to identify shapes (e.g., if a square is “tilted,” it is no longer a square). • Students may not use relevant attributes to identify shapes (e.g., a square may consist of curved segments).
Analytic	<ul style="list-style-type: none"> • Students begin to focus on the necessary properties of shapes (e.g., a square has four right angles and four sides the same length). • Subjective groupings of shapes are used instead of conventional ones (e.g., rejection of the idea that a square is a special type of rectangle). • Students do not appreciate proof. They might think that geometric theorems can be established as true by evidence from a number of examples.
Abstract	<ul style="list-style-type: none"> • Students can form definitions for shapes and express them. • Students can understand conventional groupings of shapes. • Students can form short chains of reasoning, perhaps linking if-then statements.
Deductive	<ul style="list-style-type: none"> • Students see the overall structure of geometry • Students recognize that formal proofs are needed to establish the truth of a conjecture. • Students understand the roles of definitions, theorems, axioms, and other tools of discourse in geometry.
Rigorous	<ul style="list-style-type: none"> • Students can engage in the study of different axiomatic systems and reason rigorously within them.

The goal of most high school geometry courses is to have students reasoning at the deductive level by the end of the year. Usiskin (1982) found, however, that many students do not show evidence of progress beyond the first two levels by the end of high school. The students in my summer school geometry course supported this research, showing little evidence of being able to reason at the deductive level. During the first week of the course, I asked students to write a proof for the item shown in **figure 1**. While some students wrote that the proof had something to do with vertical angles, none was able to link chains of reasoning to begin to prove the statement deductively. In addition, some students exhibited confusion about identifying sides and angles in triangles. When asked to identify the largest angle in a triangle, students would identify the longest side instead. Little evi-

dence of thinking beyond the lowest two van Hiele levels was evident as the class began.

Van Hiele theory suggests a means for sequencing instruction to help teachers achieve the goal of moving students from one level to the next. According to van Hiele theory, instructional units should contain five phases: information, guided orientation, explication, free orientation, and integration (Clements 2003). During the first phase (information), teachers introduce the vocabulary and concepts necessary for completing a task. During the second phase (guided orientation), students are engaged with the concepts in order to begin to develop understandings of them and the connections among them. Students are to verbalize their understandings of the concepts and their connections during the third phase (explication). During the fourth phase (free orientation), students should be challenged to solve problems related to the concepts at hand and make connections among them. Finally, during the fifth phase (integration), teachers help students reflect upon the observations they have made and begin to understand the overall structure of the concepts and where those structures fit in the scheme of formal mathematics. An instructional unit can be structured to revisit each of the five phases as necessary.

I applied the idea of learning phases in two different situations: first, when teaching students the relationships among different types of quadrilaterals; and second, in teaching the triangle inequality.

LEARNING PHASES EXAMPLE 1: RELATIONSHIPS AMONG QUADRILATERALS

According to NCTM (2000), “Students should enter high school understanding the properties of, and relationships among, basic geometric objects” (p. 310). My students had problems understanding the relationships among different types of quadrilaterals. At one point, I drew a square and a rectangle on the board and asked the class whether or not a square could be considered a rectangle. When none of the students could explain why a square is a type of rectangle, I decided to pay some serious attention to helping them understand the relationships that exist among different types of quadrilaterals. I organized my instructional approach for the topic around the five phases prescribed by van Hiele theory.

During the first learning phase, I used two different methods to help students gain information. First, students found the following terms in their textbook and wrote out the definitions: *parallelogram*, *rhombus*, *rectangle*, *square*, *trapezoid*, *isosceles trapezoid*, and *quadrilateral*. Second, I drew one example of each object on the chalkboard. The following phases would involve processing and connecting the various

Suppose that we know $\angle OUT$ and $\angle OTU$ in the following diagram are congruent. Explain how we know that $\angle SUQ$ and $\angle RTP$ are congruent:

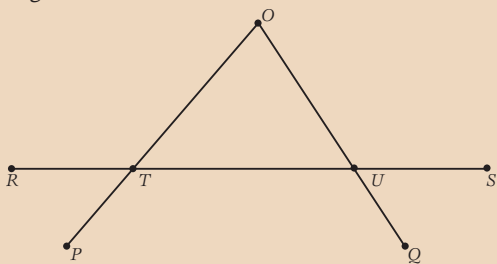


Fig. 1 Proof item given after first week of the course

bits of information introduced in phase 1.

During phase 2 (guided orientation), students produced as many examples as possible of each type of quadrilateral they had defined. After working individually, students were asked to add more examples of each type of quadrilateral to those already on the chalkboard until they could think of no more examples for each category. By the end of this phase, we had a board full of quadrilaterals.

Students now could begin investigating the fundamental characteristics of each shape family. In phase 3 (explication), students formed definitions for each shape family. I asked such questions as, What makes something a rectangle? What makes something a square? and What makes something a trapezoid? In answering these questions, my students referred, not to the definitions they had copied from the textbook, but to the examples they had produced on the board. This response indicated to me that they had at best a shallow understanding of the definitions they had copied and that the true internalization of the definitions occurred only as we progressed through the learning phases.

In the next phase (free orientation), I posed problems that challenged students to extend and deepen their understanding of the different types of quadrilaterals and the relationships among them. I gave students several true/false questions that required them to make connections among the definitions they had begun to understand during the previous phase. For example, I asked them to decide (and to justify) whether a parallelogram is a type of quadrilateral, a rectangle is a type of trapezoid, and a rectangle is a type of parallelogram.

After our discussion of the questions, we moved on to phase 5, using Venn diagrams to show the relationships among the quadrilaterals. This activity helped students reflect on the content they had explored and see the overall structure.

The five-phase framework helped me design instruction that moved students from having little understanding of relationships among quadrilaterals

to seeing and reflecting on some of those relationships. By attending to each of the first four phases, I gave my students the background needed to understand the abstract relationships depicted in the Venn diagram introduced in phase 5.

LEARNING PHASES EXAMPLE 2: THE TRIANGLE INEQUALITY

I also used the five learning phases to explore triangle inequality. The information phase consisted of class discussions on differentiating between sides and angles in a triangle. At first, some students were not able to perform this task. The class discussed the difference between a side and an angle in a triangle, and I introduced the formal notation for indicating sides and angles. These discussions took place on several occasions before we did the activities described in the remainder of this section.

To begin phase 2 (guided orientation), I gave each student a piece of uncooked spaghetti and asked them to break it in any two places. I then asked them to arrange the pieces (representing three line segments) into a triangle. Initially, the students broke the spaghetti to produce pieces close to the same length. Hence, all of them were able to create triangles with their pieces. I handed out another piece of spaghetti to each student and asked them to try to break the spaghetti so that the pieces formed would *not* make up a triangle. About half of my students were able to accomplish this task during the time allotted.

During phase 3 (explication), students shared their strategies for breaking the spaghetti into segments that did not form a triangle. I asked them to hold up the segments they had made with the second piece of spaghetti and to talk about any relationships that were apparent among the segment lengths. Students were able to verbalize that one of the segments was longer than the other two put together. I then asked them to compare the lengths of the segments from the first piece of spaghetti, which did work to form a triangle. Here, they noticed that any two of the segments put together were longer than the third one. This indicated to me that, as a class, we had arrived at an informal understanding of the triangle inequality.

In phase 4 (free orientation), I gave my students lengths of segments produced in several hypothetical spaghetti-breaking scenarios. Their task was to figure out whether or not the segment lengths could form a triangle. Students were quite successful with the problems, referring to their own experience as they solved each one. The fact that they had already “acted out” similar situations using concrete objects made the task manageable for each of them.

During phase 5, students formalized their knowledge of the triangle inequality. In a class dis-

cussion, we revisited one of the hypothetical spaghetti-breaking scenarios, in which the pieces had lengths of 6 cm, 4 cm, and 9 cm. I altered the situation a bit, asking what the possible side lengths for the third piece could be if we knew only that the first side length was 6 cm and the second was 4 cm. The class decided that the side would have to be more than 2 cm long and less than 10 cm long. I took this opportunity to show the class what this statement would look like in formal symbolic terms, labeling the first side a , the second side b , and the third side c (producing $2 < c < 10$). I then pointed out that we know this because we already decided earlier that any two sides added together produce a segment longer than the third. This discussion gave me the opportunity to introduce a formal statement that connected to the informal notion they already had developed: In any triangle with side length measurements a , b , and c , it is true that $a + b > c$, $b + c > a$, and $a + c > b$.

The five phases for the triangle inequality bear some resemblance to the five phases for the relationships among quadrilaterals. In both scenarios, formal mathematical tools for expressing geometric relationships were introduced only after students had constructed concrete meanings for the geometric objects involved. I believe that I would have done my students a disservice if I had started with formalisms without first helping the students develop the background to understand them.

CONCLUSION

Was the use of the five learning phases worthwhile? On the final exam, most students successfully answered true/false statements regarding quadrilateral relationships, and they retained parts of our discussion of the triangle inequality. Their responses to the triangle inequality question (see **fig. 2**) helped me reflect on how to refine my approach to the topic in future sessions. The class was almost evenly split between choices b and d in the question. Only one student selected choice c. In choice b, two of the sides shown add up to be the exact length of the third side, while in choice d the sides fit the criteria of the triangle inequality. Choices a and c each have two side lengths that add to be less than the length of the third side. The performance on this item indicated to me that the problems used in phase 4 of my instruction should have probably included more scenarios in which two of the side lengths added up to be exactly the length of the third side.

My use of the five-phase approach helped my students move toward the abstract level of thinking about relationships among quadrilaterals and to develop and formalize their ideas about the triangle inequality. In addition, the five phases gave me a framework for reflecting upon my instruction of

Determine which set of numbers can be the measurements of the sides of a triangle (not necessarily a right triangle).

- a. 2, 6, 3
- b. 3, 10, 13
- c. 4, 6, 1
- d. 5.1, 7, 2.3

Fig. 2 Final exam question on the triangle inequality

the triangle inequality, identifying weaknesses in it, and deciding upon a course of action for future instruction. Van Hiele theory may or may not provide a method for serving the “entire meal” of geometry to your students, but the use of the five-phase framework can still result in positive learning outcomes for students and teachers alike.

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